Determining the mass of a double star

Herbert E. Müller, January 2017, herbert-mueller.info

Let A and B be the two components of a binary star, e.g. α Centauri A&B, see fig. 1 below. The vector \( \vec{AB} \) moves on an ellipse in space. If one can measure the period of revolution \( T \), the distance \( d \) from Earth, and the largest angular diameter \( 2a \) of the upright ellipse, then one can afterwards calculate the overall mass \( M \) of the binary star, using Keplers 3\textsuperscript{rd} law \( GM=(2\pi/T)^2(da)^3 \). If \( T \) is measured in years and \( d \) is measured in AU (= ly+63240), then \( M=(da)^3/T^2 \) is the binary star's mass in units of our sun's mass.

The problem is: the ellipse is usually not at right angles to the line of sight, rather it is tilted by an angle called inclination \( i \), and all angular diameters (except the one on the axis of tilt) appear shortened by perspective. The tilted ellipse has apparent main axes \( \hat{a} \) and \( \hat{b} \), and the projected focal point is located at coordinates \( (f_x,f_y) \) with respect to the apparent main axes. (α Centauri: \( T = 79.9 \) a, \( d = 4.27 \) ly, \( \hat{a} = 16.02'' \), \( \hat{b} = 3.07'' \), \( f_x = +5.67'' \), \( f_y = +1.13'' \).)

The following is about retrieving the true main axes \( a \) and \( b \) from these observations.

We choose the \( x,y,z \)-coordinate system with the origin \( O \) in the center of the ellipse, and the \( z \)-axis along the line of sight from Earth to this center. The star A is located in the focal point \( F \) of the ellipse, and the star B is moving on the ellipse. From now on we use \( F: (f_x,f_y,f_z) \) for the focal point (the star A), \( P: (x,y,z) \) for the moving ellipse point (the star B), and \( A: (a_x,a_y,a_z) \) and \( B: (b_x,b_y,b_z) \) for the endpoints of the two main axes. The focal vector is \( \vec{OF}=\vec{r} \), the ellipse vector is \( \vec{OP}\equiv\vec{OF}+\vec{FP}=\vec{r} \), and the main axes vectors are \( \vec{OA}=\hat{a} \) and \( \vec{OB}=\hat{b} \).

The projection of an ellipse has some interesting features:

1. The \( x,y \)-projection of a spatial ellipse \( E \) centered at \( O \) is an ellipse \( \tilde{E} \) centered at \( O \).
2. The projections of \( F, A \) and \( B \) do not coincide with \( \tilde{F}, \tilde{A} \) and \( \tilde{B} \) (the focal point and main axes end points of \( \tilde{E} \)) . The projections of \( \vec{OA} \) and \( \vec{OB} \) are conjugated axes of \( \tilde{E} \).
3. Keplers 2\textsuperscript{nd} law of constant area velocity \( \vec{FP}\times d\vec{FP}/dt \) is also valid for the projected area.

Photos of the double star taken at regular time intervals give us \( x,y \)-projections of the vector \( \vec{FP} \), allowing us to construct the projected ellipse, see fig. 2 below. Because of point 3, the time schedule of the point \( P \) is not important except for the period \( T \), and we parametrise the ellipse with \( \tilde{r}=\tilde{a}\cos(t)+\tilde{b}\sin(t) \); here \( t \) is NOT time, but simply a parameter in \([0,2\pi]\).

**Problem:**

Given the projected ellipse \( \tilde{E} \), i.e. the apparent main axes \( \vec{OA} \) and \( \vec{OB} \) with lengths \( \hat{a} \) and \( \hat{b} \), and the projected (true) focal vector \( (f_x,f_y) \), determine the spatial ellipse \( E \), i.e. its main axes \( \vec{OA}=\hat{a} \) and \( \vec{OB}=\hat{b} \), the missing focal coordinate \( f_z \), and the rotation with axis \( d \) and angle \( i \) (inclination) that turns the ellipse upright.
Solution:

We orient the x,y-coordinate axes along the apparent main axes; the coordinates of $\tilde{A}$ and $\tilde{B}$ are $(\tilde{a},0)$ and $(0,\tilde{b})$. The true main axes $\tilde{a}$ and $\tilde{b}$ are obtained by rotation. The 3 points $O$, $F$, $A$ are in line, and the eccentricity of the spatial ellipse is $e=\mathcal{O}F/\mathcal{O}A$.

$$\tilde{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \tilde{a} \\ 0 \\ \alpha \end{pmatrix} \cos(\tau)+\begin{pmatrix} 0 \\ \tilde{b} \\ \beta \end{pmatrix} \sin(\tau) \quad \text{and} \quad \tilde{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} \tilde{a} \\ 0 \\ \alpha \end{pmatrix} (-\sin(\tau))+\begin{pmatrix} 0 \\ \tilde{b} \\ \beta \end{pmatrix} \cos(\tau) \quad \text{and} \quad \tilde{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = e \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}.$$ 

Here we added two missing coordinates $\alpha$ and $\beta$ (they will be needed later on).

Let $\xi \equiv f_x/\tilde{a}$ and $\eta \equiv f_y/\tilde{b}$, then $e=\sqrt{\xi^2+\eta^2}$. Combining the 1$^{\text{st}}$ and 3$^{\text{rd}}$ eqn. above gives $\cos(\tau) = \xi/e$ and $\sin(\tau) = \eta/e$, i.e. $\tau = \arctan(\eta/\xi)$. Inserting above we obtain

$$\tilde{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \tilde{a} \xi/e \\ \tilde{b} \eta/e \\ a_z \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} -\tilde{a} \eta/e \\ \tilde{b} \xi/e \\ b_z \end{pmatrix} \quad \text{and} \quad \tilde{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \tilde{a} \xi \\ \tilde{b} \eta \\ f_z e \end{pmatrix}.$$ 

The two main axes are perpendicular to each other: $0=\tilde{a} \cdot \tilde{b} = a_x b_z-(\tilde{a}^2-\tilde{b}^2)\xi \eta/e^2$, and therefore $a_x b_z=(\tilde{a}^2-\tilde{b}^2)\xi \eta/e^2(*)$.

Now set $A=(a_x^2+a_y^2)/(1-e^2)$ and $B=(b_x^2+b_y^2)$ and $S=-(a_x b_x+a_y b_y)\sqrt{1-e^2}$, or, in terms of known quantities, $A=(\tilde{a}^2\xi^2+\tilde{b}^2\eta^2)/(1-e^2)$ and $B=(\tilde{a}^2\eta^2+\tilde{b}^2\xi^2)/e^2$ and $S=(\tilde{a}^2-\tilde{b}^2)\xi \eta \sqrt{1-e^2}/e^2$.

Eqn. (*) becomes $a_x b_z \sqrt{1-e^2}=S$. With $b^2=a^2(1-e^2)=A+a_x^2(1-e^2)$ and $b^2=B+b_x^2$, we obtain after squaring a biquadratic eqn. for $b$: $|b^2-A||b^2-B|=S^2$.

The solution is $b=\sqrt{(A+B)+\sqrt{(A-B)^2+4S^2}}$ and then follows $a=b \div \sqrt{1-e^2}$.

The biquadratic eqn. for $b_z$ is $|b_z^2-A+B|^2=S^2$.

The solution is $b_z=\pm \sqrt{(A-B)+\sqrt{(A-B)^2+4S^2}}$ and then follows $a_z=S \div (b_z \sqrt{1-e^2})$.

The plane of the ellipse $\tilde{E}$ intersects the $x,y$-plane in an axis $d$ going through $O$ and forming an angle $\phi$ (or ellipse angle $t_d$) with the $x$-axis. $E$ can be flipped into the $x,y$-plane with a rotation by an angle $i$ around $d$.

The area of $E$ is $\pi a b$, and the area of $\tilde{E}$ is $\pi \tilde{a} \tilde{b}$. The ratio of the two areas is $(\tilde{a}\tilde{b}) / (a b) = \cos i$, i.e. $i=\arccos((\tilde{a}\tilde{b}) / (a b))$.

To determine the axis of rotation we need $\alpha$ and $\beta$. Comparing the 4 eqn.s for $\tilde{a}$ and $\tilde{b}$ we find

$$\begin{aligned} a_x &= \frac{1}{e} \left( \frac{\xi}{\eta} \right) \frac{\alpha}{\beta} \xi \\ b_z &= \frac{1}{e} \left( \frac{\xi}{\eta} \right) \frac{\alpha}{\beta} \frac{\eta}{\xi} \\ \alpha &= \frac{1}{e} \left( \frac{\xi}{\eta} \right) \frac{\alpha}{\beta} \xi \\ \beta &= \frac{1}{e} \left( \frac{\xi}{\eta} \right) \frac{\alpha}{\beta} \eta \end{aligned}$$

The two lines of this eqn. written separately are $\alpha=(\xi a_z-\eta b_z)/e$ and $\beta=(\eta a_z+\xi b_z)/e$.

The first of the two points $d \cap \tilde{E}$ has the z-coordinate $\alpha \cos(t_d)+\beta \sin(t_d)=0$, so $\tan(t_d)=-\alpha/\beta$.

Therefore $\cos(t_d)=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$ and $\sin(t_d)=\frac{-\alpha}{\sqrt{\alpha^2+\beta^2}}$.

The two points $d \cap E$ are now $\pm (\tilde{a} \cos(t_d), \tilde{b} \sin(t_d),0)$ or $\pm (\tilde{a} \beta, -\tilde{b} \alpha, 0) \div \sqrt{\alpha^2+\beta^2}$.
The angle between the axis of rotation $d$ and the x-axis is given by \( \tan \phi = -\left( \frac{\hat{b} \alpha}{\hat{a} \beta} \right) \).

Therefore \( \cos \phi = \frac{\hat{a} \beta}{\sqrt{\hat{a}^2 + \hat{b}^2}} \) and \( \sin \phi = \frac{-\hat{b} \alpha}{\sqrt{\hat{a}^2 + \hat{b}^2}} \).

The x,y-part of the transformation that flips the ellipse $E$ upright is

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  \cos \phi & -\sin \phi \\
  \sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  0 & \cos^{-1} \phi
\end{pmatrix} \begin{pmatrix}
  \cos \phi & -\sin \phi \\
  \sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix} + \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix} \begin{pmatrix}
  \sin \phi & -\sin \phi \cos \phi \\
  -\sin \phi \cos \phi & \cos^2 \phi
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix}.
\]

Using the previous formulas, one can write this as

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  x \\
  y
\end{pmatrix} + \frac{ab}{\sqrt{\hat{a}^2 + \hat{b}^2}} \begin{pmatrix}
  \hat{b}/\hat{a} & \hat{a}/\hat{b} \\
  \hat{a}/\hat{b} & \hat{b}/\hat{a}
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix}.
\]

Pictures of Alpha Centauri AB:

**Figure 1a&b:** α Centauri A (left) and B (right).

2015-05-04, taken with VisAO / MagAO / University of Arizona. [https://visao.as.arizona.edu/wp-content/uploads/2015/05/magao_visual.jpg](https://visao.as.arizona.edu/wp-content/uploads/2015/05/magao_visual.jpg)


Analysis for Alpha Centauri AB:

**Figure 2a:** The projected ellipse of α Centauri AB is reconstructed from 4 star positions in the years 2000.5:2015 (star B is moving FLTR).*

$d = 4.37 \text{ ly}$, $T = 79.9 \text{ a}$, $\hat{a} = 16.02''$, $\hat{b} = 3.07''$, $f_x = +5.67''$, $f_y = +1.13''$

*Based on a figure from en.wikipedia.org/wiki/Alpha_Centauri.

Unfortunately the 4 star positions are not measured, but calculated from orbital parameters.

**Figure 2b:** The upright ellipse.

My analysis: $a = 17.43''$, $b = 14.99''$, $e = 0.511$, $i = 79.15^\circ$, $M = 1.999 \text{ M(Sun)}$.

English wikipedia: $a = 17.57''$, $b = 15.03''$, $e = 0.518$, $i = 79.20^\circ$, $M = 2.007 \text{ M(Sun)}$. 
clear
T=79.9; d=4.37; at=16.02; bt=3.07; fx=5.67; fy=1.13;
t=0:710; t=t/113; x=at*cos(t); y=bt*sin(t);

% projected ellipse
figure(1);clf;
plot(x,y,'k','linewidth',1,fx,fy,'ko');hold on;
plot([-at,at],[0,0],'-','linewidth',1,[-bt,bt],'-','linewidth',1);

% projected main axes
xi=fx/at;eta=fy/bt; e=sqrt(xi^2+eta^2);
ax=fx/e;ay=fy/e; bx=-eta*at/e;by=xi*bt/e;
plot([-ax,ax],[-ay,ay],'r-','linewidth',1,-[bx,bx],[-by,by],'b-','linewidth',1)
hold off; axis([-1 1 -1 1]*20);axis equal;
title(['at=' num2str(at) ', bt=' num2str(bt) ',f_x=' num2str(fx) ', f_y=' num2str(fy) '.'])

% main axes and missing z-coordinates
A=((at*xi)^2+(bt*eta)^2)*(1-e^2)/e^2;
B=((-eta*at)^2+(xi*bt)^2)/e^2;
S=(at^2-bt^2)*xi*eta*sqrt(1-e^2)/e^2;
b=sqrt( (A+B)/2+sqrt((A-B)^2/4+S^2) );a=b/sqrt(1-e^2);
bz=sqrt( (A-B)/2+sqrt((A-B)^2/4+S^2) ); az=S/sqrt(1-e^2)/bz;
ci=at*bt/a/id=acos(ci)*180/pi;
alp=(xi*az-eta*bz)/e;bet=(eta*az+xi*bz)/e;

% upright ellipse
xp=x+(a*b-at*bt)/(bt^2*alp^2+at^2*bet^2)*alp*(bt/at*alp*x+alp*y);
yp=y+(a*b-at*bt)/(bt^2*alp^2+at^2*bet^2)*bet*(alp*x+at/bt*bet*y);

figure(2);clf;
plot(xp,yp,'k','linewidth',1,fxp,fyp,'ro');hold on;
plot([-axp,axp],[-ayp,ayp],'-','linewidth',1,-[bxp,bxp],[-byp,byp],'-','linewidth',1)
plot(x,y,'-',...,'linewidth',1,'color',[1 1 1])*0.5);
plot(fx,fx,'o','markeredgecolor',[1 0.5 0.5]);
plot([-ax,ax],[-ay,ay],'-','linewidth',1,'color',[0.5 0.5 1]);
hold off; axis([-1 1 -1 1]*20);axis equal;
title(['at=' num2str(at) ', bt=' num2str(bt) ', a=' num2str(a) ', b=' num2str(b) '.'])

% numerical analysis
display(['a = ' num2str(a) ', b = ' num2str(b) ', e = ' num2str(e) ', i = ' num2str(id) ]); display(['d = ' num2str(d) ' ly, M/M(Sun) = ' num2str((d*63240*a*4.85E-6)^3/T^2) '.'])

% printing
figure(1); set(gcf,"paperposition",[0. 0. 5. 5.]); print gcf fig1.png;
figure(2); set(gcf,"paperposition",[0. 0. 5. 5.]); print gcf fig2.png;